Rewriting Graphically with Cartesian Traced Categories

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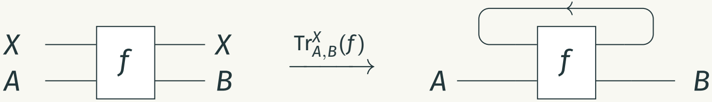
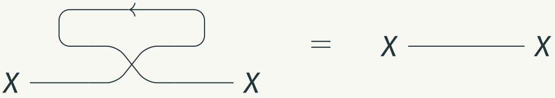
Symmetric traced monoidal categories



Symmetric traced monoidal category

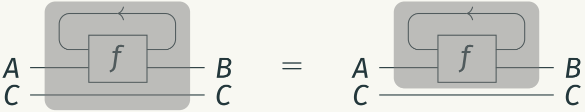
Tightening

Yanking

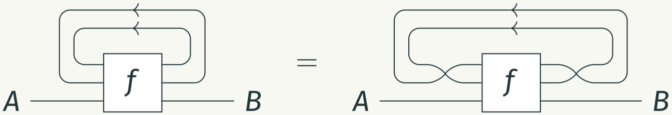


Symmetric traced monoidal category

Superposing



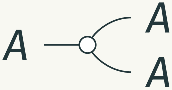
Exchange



Cartesian categories

The tensor is a product and the unit object is terminal.

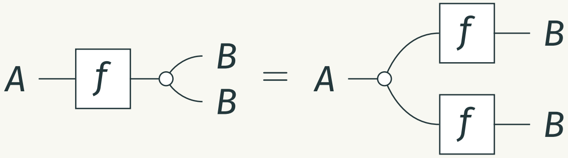
$$\Delta_A : A \to A \otimes A$$

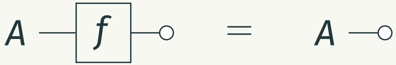


$$\Diamond_A : A \to I$$

Cartesian categories – axioms

Naturality

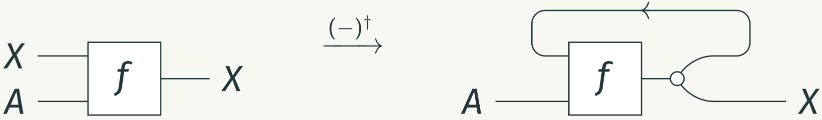




among others...

Cartesian traced categories

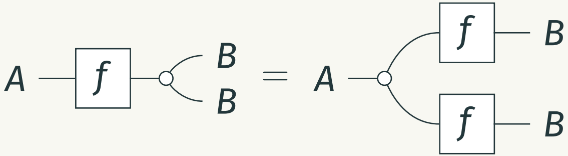
Product + trace = fixpoint operator (Hasegawa 1997)



Also known as dataflow categories.

Graphical languages for Cartesian categories

Applying Cartesian axioms require a rewriting of the graph



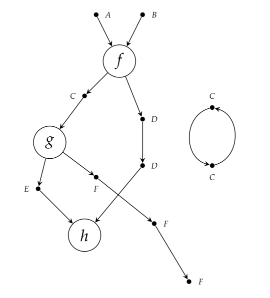
'Only connectivity matters' no longer applies!

Combinatorial graph language required

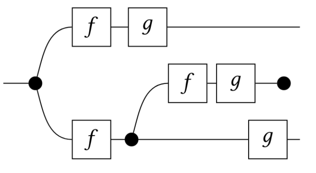
Graphical languages for Cartesian categories

Can we use something off the shelf?

String graphs (Dixon, Kissinger)



Hypergraphs (Bonchi, Gadduchi, Kissinger, Sobociński, Zanasi)



The compact closed problem

These frameworks are based in compact closed categories.

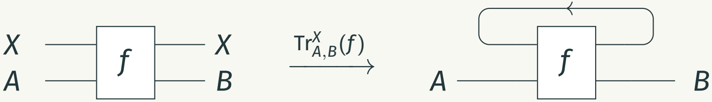
It is possible to construct a trace using the compact closed structure.

But finite products become biproducts in a compact closed category.

And if we add a Cartesian structure to a compact closed category it becomes trivial anyway.

The compact closed problem

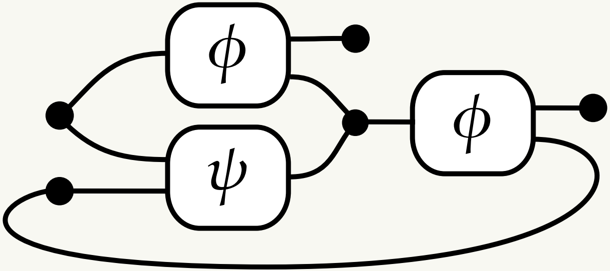
The trace must be constructed as an atomic operation.



Goal: define a sound and complete graph language for STMCs with atomic trace.

Hypergraphs

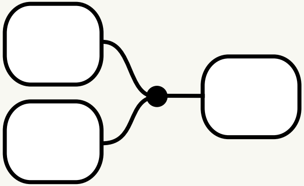
‘Vanilla’ hypergraphs



Definition

$\mathbf{Hyp}_\Sigma$ is the category with objects the labelled hypergraphs over a signature Σ and morphisms the labelled hypergraph homomorphisms.

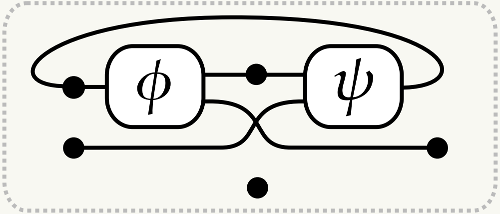
Hypergraphs are not enough



not allowed!

We could rule out the ones that don’t fit our criteria, but this might not be compositional.

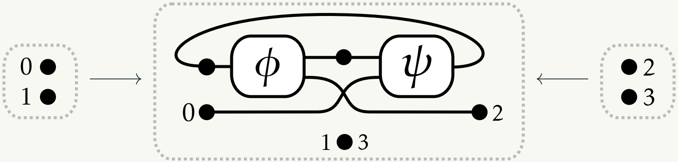
Linear hypergraphs



Definition

LHyp$_\Sigma$ is the category with objects the linear hypergraphs labelled over a signature Σ and morphisms the labelled linear hypergraph homomorphisms.

Cospans of linear hypergraphs



A cospan $M \to H \leftarrow N$ is discrete if M and N contain no edges.

An monogamous cospan only picks the 'open' vertices.

Definition

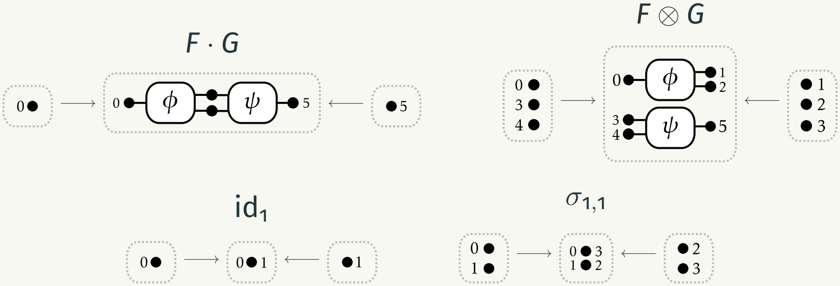
$MCsp_D(\mathbf{LHyp}_\Sigma)$ is the category of monogamous cospans over $\mathbf{LHyp}_\Sigma$.

A graph language for STMCs

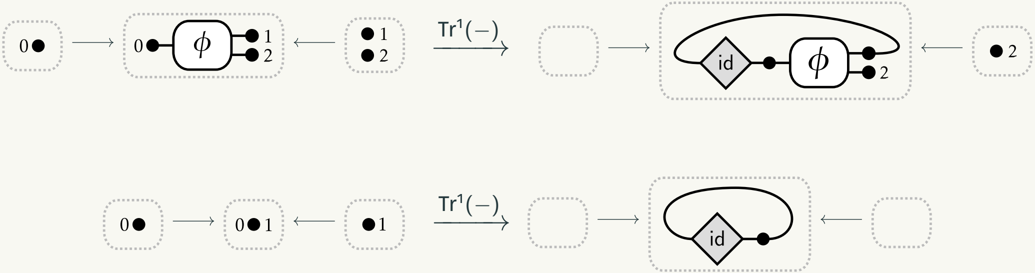
Are linear hypergraphs a suitable graph language for STMCs?

We need to define the operations of an STMC.

Most are fairly obvious...

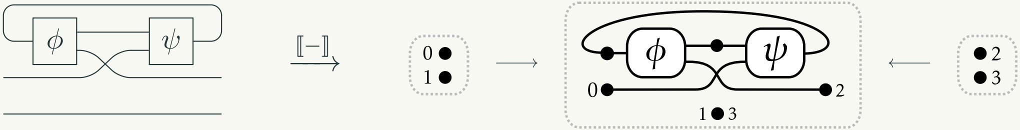


Trace



We fix a traced PROP $\mathbf{Term}_\Sigma$ generated over some signature Σ.

$$[\![-]\!] \;:\; \mathbf{Term}_\Sigma \;\to\; \mathit{MCsp}_D(\mathbf{LHyp}_\Sigma)$$



Soundness

Equal terms in the category $\Rightarrow$ Isomorphic interpretations as hypergraphs

Soundness

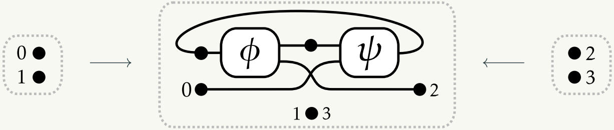
Theorem (Soundness)

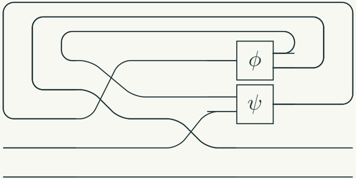
For any morphisms $f, g \in \textbf{Term}_{\Sigma}$, if $f = g$ under the equational theory of the category then their interpretations as cospans of labelled linear hypergraphs are isomorphic, $[\![f]\!] \equiv [\![g]\!]$

Definability

A cospan of labelled linear hypergraphs $\Rightarrow$ A set of corresponding terms in the category

Definability





$$\mathrm{Tr}^3(\sigma_{2,1} \otimes \mathrm{id}_2 \cdot \mathrm{id}_2 \otimes \sigma_{1,1} \otimes \mathrm{id}_1 \cdot \phi \otimes \psi \otimes \mathrm{id}_2)$$

Definability

$$\langle\!\langle - \rangle\!\rangle \ : \ MCsp_D(\mathbf{LHyp}_\Sigma) \ \to \ \mathbf{Term}_\Sigma$$

Proposition (Definability)

For any $F \in \mathbf{LHyp}_\Sigma$ and edge order $\leq$, then $m \to F \leftarrow n \equiv [\![\langle\!\langle m \to F \leftarrow n \rangle\!\rangle_{\leq}]\!]$.

Coherence

But we cannot conclude completeness yet!

A labelled linear hypergraph $\Rightarrow$ **Unique** morphism in the category, **up to the equational theory**

Proposition (Coherence)

For all orderings of edges $\leq_x$ on some $F \in \mathbf{LHyp}_\Sigma$,

$$\langle\!\langle m \to F \leftarrow n \rangle\!\rangle_{\leq_1} = \langle\!\langle m \to F \leftarrow n \rangle\!\rangle_{\leq_2} = \cdots = \langle\!\langle m \to F \leftarrow n \rangle\!\rangle_{\leq_x}$$

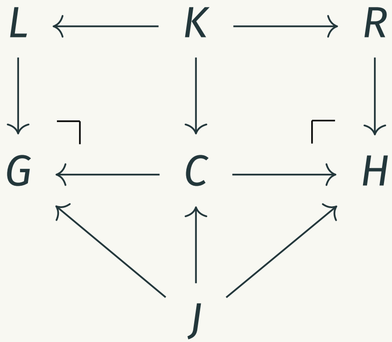
Completeness

Theorem (Completeness)

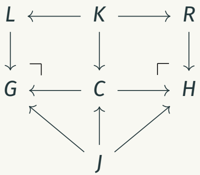
For any cospan of linear hypergraphs $m \to F \leftarrow n \in MCsp_D(\mathbf{LHyp}_\Sigma)$ *there exists a unique morphism* $f \in \mathbf{Term}_\Sigma$*, up to the equations of the STMC, such that* $[\![f]\!] = F$*. Moreover, for any* $f \in \mathbf{Term}_\Sigma$, $\langle\langle [\![f]\!] \rangle\rangle = f$.

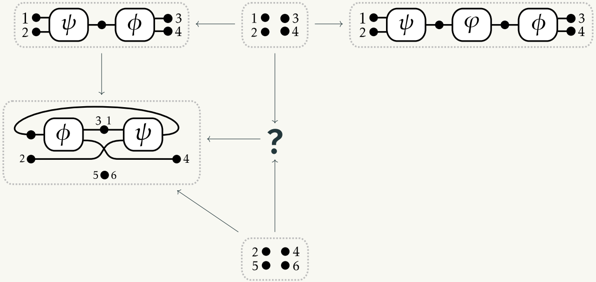
Graph rewriting

DPO rewriting



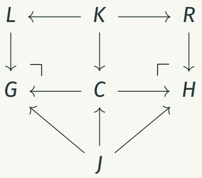
DPO rewriting





We need a guarantee that this pushout complement is unique.

Adhesive categories



In an adhesive category, if we have

- a rewrite rule $L \overset{p}{\leftarrow} K \rightarrow R$ where p is mono,
- a matching $L \rightarrow G$

then the pushout complement $K \rightarrow C \rightarrow R$ is unique (if it exists).

We have already met an adhesive category:

Proposition

Hyp$_\Sigma$ *is an adhesive category.*

Partial aAdhesive categories

Unfortunately $\mathbf{LHyp}_{\Sigma}$ is not adhesive.

Definition (Partial adhesive categories (Kissinger))

A category $\mathcal{P}$ is called a partial adhesive category if it is a full subcategory of an adhesive category $\mathcal{A}$ and the inclusion functor $I : \mathcal{P} \to \mathcal{A}$ preserves monomorphisms.

Partial adhesive categories

Proposition

$\mathbf{LHyp}_\Sigma$ is a full subcategory of $\mathbf{Hyp}_\Sigma$.

Proposition

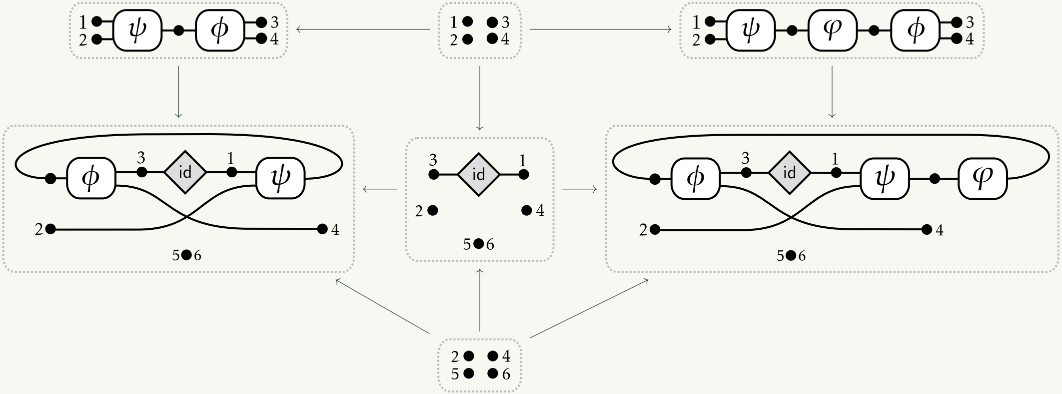
The inclusion functor $I : \mathbf{LHyp}_\Sigma \to \mathbf{Hyp}_\Sigma$ preserves monomorphisms.

Corollary

$\mathbf{LHyp}_\Sigma$ is a partial adhesive category.

So for matchings that are mono, graph rewriting is well-defined.

DPO rewriting example



Conclusion

- Sound and complete graph language for symmetric traced monoidal categories with a Cartesian structure
- This is by defining the trace as an atomic operation
- Linear hypergraphs form a partial adhesive category
- So graph rewriting can be performed as with regular hypergraphs!